



TITLE:

# Stability of Delaunay surfaces as steady states for a geometric evolution equation (Singularity theory of differential maps and its applications)

AUTHOR(S):

Kohsaka, Yoshihito

---

CITATION:

Kohsaka, Yoshihito. Stability of Delaunay surfaces as steady states for a geometric evolution equation (Singularity theory of differential maps and its applications). 数理解析研究所講究録 2017, 2049: 57-70

ISSUE DATE:

2017-10

URL:

<http://hdl.handle.net/2433/237052>

RIGHT:

# Stability of Delaunay surfaces as steady states for a geometric evolution equation

Yoshihito Kohsaka

Graduate School of Maritime Sciences, Kobe University

## 1 Introduction

Let  $\Gamma_t \subset \mathbb{R}^3$  be a evolving surface with respect to time  $t$ . The surface diffusion equation

$$V = -\Delta_{\Gamma_t} H \text{ on } \Gamma_t \quad (1)$$

is one of the geometric evolution laws, where  $V$  is the normal velocity of  $\Gamma_t$ ,  $H$  is the mean curvature of  $\Gamma_t$ , and  $\Delta_{\Gamma_t}$  is the Laplace-Beltrami operator on  $\Gamma_t$ . In our sign convention, the mean curvature  $H$  for spheres with outer unit normal is negative.

The mean curvature flow

$$V = H \text{ on } \Gamma_t \quad (2)$$

is a well-known geometric law and represented as the  $L^2$ -gradient flow for the area functional of  $\Gamma_t$ . This implies a variational structure that the area of the surface  $\Gamma_t$  decreases with respect to time  $t$ . On the other hand, the surface diffusion equation (1) is the  $H^{-1}$ -gradient flow for the area functional of  $\Gamma_t$ , so that this geometric evolution equation has a variational structure that the area of the surface  $\Gamma_t$  decreases with respect to time  $t$  whereas the volume of the region enclosed by the surface  $\Gamma_t$  is preserved.

In this paper, we consider the following problem. For  $\phi_{\pm} : \mathbb{R}_+ \rightarrow \mathbb{R}$ , set

$$\begin{aligned} \Pi_{\pm} &= \{(\phi_{\pm}(|\boldsymbol{\eta}|), \boldsymbol{\eta})^T \mid \boldsymbol{\eta} \in \mathbb{R}^2\}, \\ \Omega &= \{(x, \boldsymbol{\eta})^T \mid \phi_{-}(|\boldsymbol{\eta}|) \leq x \leq \phi_{+}(|\boldsymbol{\eta}|), \boldsymbol{\eta} \in \mathbb{R}^2\}, \\ \partial\Omega &= \Pi_{-} \cup \Pi_{+}. \end{aligned}$$

Note that  $\Pi_{\pm}$  are the axisymmetric surfaces. Let us assume that  $\Gamma_t \subset \Omega$  and the motion of  $\Gamma_t$  is governed by

$$\begin{cases} V = -\Delta_{\Gamma_t} H \text{ on } \Gamma_t, \\ (N_{\Gamma_t}, N_{\Pi_{\pm}})_{\mathbb{R}^3} = \cos \theta_{\pm} \text{ on } \Gamma_t \cap \Pi_{\pm}, \\ (\nabla_{\Gamma_t} H, \nu_{\pm})_{\mathbb{R}^3} = 0 \text{ on } \Gamma_t \cap \Pi_{\pm}, \\ \Gamma_t|_{t=0} = \Gamma_0. \end{cases} \quad (3)$$

Here,  $N_{\Gamma_t}$  and  $N_{\Pi_{\pm}}$  are the outer unit normals to  $\Gamma_t$  and  $\Pi_{\pm}$ , respectively, and  $\nu_{\pm}$  are the outer unit co-normals to  $\partial\Gamma_t$  on  $\Gamma_t \cap \Pi_{\pm}$ .

Let  $\Gamma_*$  be the steady states for (3) and  $H_*$  be the mean curvature of  $\Gamma_*$ . Then  $\Gamma_*$  satisfies

$$\begin{cases} \Delta_{\Gamma_*} H_* = 0 & \text{on } \Gamma_*, \\ (\nabla_{\Gamma_*} H_*, \nu_{\pm})_{\mathbb{R}^3} = 0 & \text{on } \Gamma_* \cap \Pi_{\pm}. \end{cases}$$

Multiplying  $H_*$  by the both side of the equation  $\Delta_{\Gamma_*} H_* = 0$  and applying the Green's formula, we obtain

$$\|\nabla_{\Gamma_*} H_*\|_{L^2(\Gamma_*)}^2 = 0.$$

Thus we see that the steady states of (3) are the constant mean curvature surfaces (CMC surfaces). In this paper, we only consider the axisymmetric CMC surfaces, which is so called the Delaunay surfaces, as the steady states  $\Gamma_*$ . For an axisymmetric perturbation from  $\Gamma_*$ , we derive the eigenvalue problem corresponding to the linearized problem for (3) and obtain the criteria of the stability of  $\Gamma_*$ .

As regards the results on the stability of the Delaunay surfaces as the variational problem for the capillary energy, we refer to Athanassenas [2], Fel and Rubinstein [6, 14], and Vogel [15, 16, 17, 18, 19]. Concerning the results on the stability as steady states for the surface diffusion equation, we refer to Abels, Garcke, and Müller [1], Depner [5], and LeCrone and Simonett [12].

## 2 The eigenvalue problem

Let  $\Gamma_*$  be a axisymmetric steady states of (3) and set

$$\Gamma_* = \{(x_*(s), y_*(s) \cos \zeta, y_*(s) \sin \zeta)^T \mid s \in [0, d], \zeta \in [0, 2\pi]\},$$

where  $s$  is the arc-length parameter of a generating curve  $(x_*(s), y_*(s))^T$ . In the following theorem, we introduce the representation formula of the Delaunay surfaces with a non-zero constant mean curvature.

**Theorem 2.1 ([9, 13])** Let  $H_*$  be a constant satisfying  $H_* \neq 0$  (assuming  $H_* < 0$ ). Then a generating curve  $(x_*(s), y_*(s))^T$  of the Delaunay surface with a constant mean curvature  $H_*$  is given by

$$\begin{cases} x_*(s) = \int_0^s \frac{1 - B \sin(2H_*(\sigma - \tau))}{\sqrt{1 + B^2 - 2B \sin(2H_*(\sigma - \tau))}} d\sigma, \\ y_*(s) = -\frac{1}{2H_*} \sqrt{1 + B^2 - 2B \sin(2H_*(s - \tau))}, \end{cases} \quad (4)$$

where  $B \geq 0$  and  $\tau \in \mathbb{R}$  are constants.

The Delaunay surface is a cylinder for  $B = 0$  (Fig. 1), an unduloid for  $0 < B < 1$  (Fig. 2), a series of spheres for  $B = 1$  (Fig. 3), and a nodoid for  $B > 1$  (Fig. 4).



Figure 1: Cylinder ( $B = 0$ )

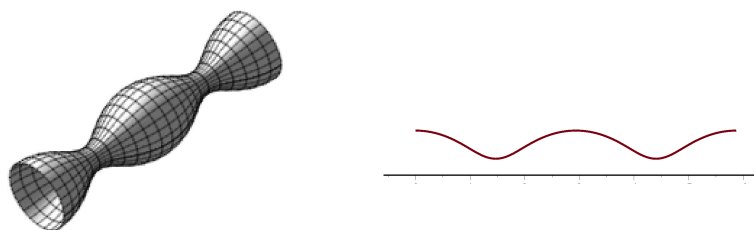


Figure 2: Unduloid ( $0 < B < 1$ )

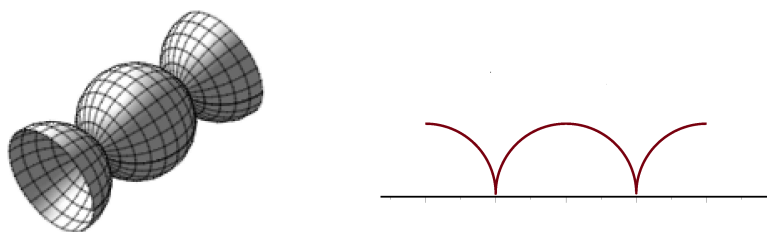
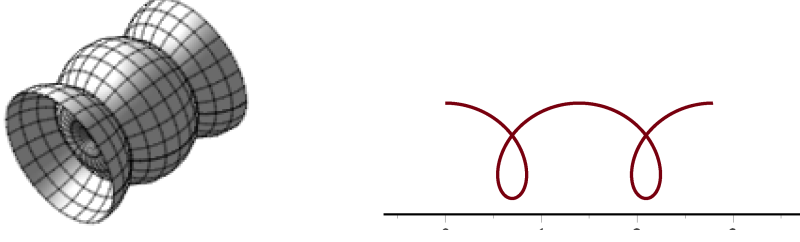


Figure 3: Series of spheres ( $B = 1$ )

Figure 4: Nodoid ( $B > 1$ )

Applying an axisymmetric perturbation  $v(s, t)$  for the Delaunay surfaces  $\Gamma_*$  and linearizing the nonlinear problem for  $v(s, t)$ , we have

$$\begin{cases} v_t = -\frac{1}{2}\Delta_{\Gamma_*}L[v] & \text{for } (s, t) \in [0, d] \times [0, T], \\ \partial_s v \pm (\kappa_{\Pi_{\pm}} \csc \theta_{\pm} - \kappa_{\Gamma_*} \cot \theta_{\pm})v = 0 & \text{for } s = 0, d, t \in [0, T], \\ \partial_s L[v] = 0 & \text{for } s = 0, d, t \in [0, T], \end{cases} \quad (5)$$

where  $L[v] = \Delta_{\Gamma_*}v + |A_*|^2v$  with

$$\Delta_{\Gamma_*} = \frac{1}{y_*} \left\{ \partial_s(y_* \partial_s) + \frac{1}{y_*} \partial_{\zeta}^2 \right\}, \quad |A_*|^2 = (-x''_* y'_* + x'_* y''_*)^2 + \left( \frac{x'_*}{y_*} \right)^2,$$

and

$$\kappa_{\Pi_{\pm}} = \pm \frac{\ddot{\phi}_{\pm}(y_*)}{\{1 + (\dot{\phi}_{\pm}(y_*))^2\}^{3/2}}, \quad \kappa_{\Gamma_*} = -x''_* y'_* + x'_* y''_*.$$

Note that  $\kappa_{\Pi_-}$  and  $\kappa_{\Pi_+}$  are the curvature of  $x = -\phi_-(y)$  at  $y = y_*(0)$  and  $x = \phi_+(y)$  at  $y = y_*(d)$ , respectively, and  $\kappa_{\Gamma_*}$  is the curvature of the generating curve  $(x_*(s), y_*(s))^T$ . Taking account of the fact that  $v$  is independent of  $\zeta$ , we have

$$\Delta_{\Gamma_*}v = \frac{1}{y_*} \{ \partial_s(y_* \partial_s v) \}.$$

For this linearized problem the corresponding eigenvalue problem is given by

$$\begin{cases} -\Delta_{\Gamma_*}L[w] = \lambda w & \text{for } s \in [0, d], \\ \partial_s w \pm (\kappa_{\Pi_{\pm}} \csc \theta_{\pm} - \kappa_{\Gamma_*} \cot \theta_{\pm})w = 0 & \text{at } s = 0, d, \\ \partial_s L[w] = 0 & \text{at } s = 0, d. \end{cases} \quad (6)$$

We say that the steady states  $\Gamma_*$  is linearly stable under an axisymmetric perturbation if and only if all of eigenvalues of (6) are negative.

Set

$$\begin{aligned}\mathcal{E} &= \left\{ w \in H^1(\Gamma_*) \mid \int_0^d w y_* ds = 0 \right\}, \\ \mathcal{X} &= \left\{ w \in (H^1(\Gamma_*))^* \mid \langle w, 1 \rangle = 0 \right\},\end{aligned}$$

where  $(H^1(\Gamma_*))^*$  is the duality space of  $H^1(\Gamma_*)$  and  $\langle \cdot, \cdot \rangle$  is the duality pairing  $(H^1(\Gamma_*))^*$  and  $H^1(\Gamma_*)$ . Also, set

$$\begin{aligned}\mathcal{D}(\mathcal{A}) &= \left\{ w \in H^3(\Gamma_*) \mid w \text{ satisfies} \right. \\ &\quad \partial_s w \pm (\kappa_{\Pi_\pm} \csc \theta_\pm - \kappa_{\Gamma_*} \cot \theta_\pm) w = 0 \text{ at } s = 0, d, \\ &\quad \left. \text{and } \int_0^d w y_* ds = 0 \right\}\end{aligned}$$

and define the linear operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{X}$  by

$$\langle \mathcal{A}w, \psi \rangle = \int_0^d \partial_s L[w] \partial_s \psi y_* ds \quad (w \in \mathcal{D}(\mathcal{A}), \psi \in \mathcal{E}).$$

Taking the symmetric bilinear form

$$\begin{aligned}I[w_1, w_2] &= \int_0^d \left\{ \partial_s w_1 \partial_s w_2 - |A_*|^2 w_1 w_2 \right\} y_* ds \\ &\quad + y_* (\kappa_{\Pi_+} \csc \theta_+ - \kappa_{\Gamma_*} \cot \theta_+) w_1 w_2 \Big|_{s=d} \\ &\quad + y_* (\kappa_{\Pi_-} \csc \theta_- - \kappa_{\Gamma_*} \cot \theta_-) w_1 w_2 \Big|_{s=0},\end{aligned}$$

and the  $H^{-1}$ -inner product

$$(w_1, w_2)_{-1} = \int_0^d \partial_s u_{w_1} \partial_s u_{w_2} y_* ds,$$

where  $u_{w_i}$  is a weak solution of

$$\begin{cases} -\Delta_{\Gamma_*} u_{w_i} = w_i & \text{for } s \in (0, d), \\ \partial_s u_{w_i} = 0 & \text{at } s = 0, d \end{cases}$$

for  $w_i \in \mathcal{X}$ , we obtain

$$(\mathcal{A}w, \psi)_{-1} = -I[w, \psi] \quad (\psi \in \mathcal{E}).$$

For the linear operator  $\mathcal{A}$  and its eigenvalues, we have the following properties.

- (P1) The operator  $\mathcal{A}$  is self-adjoint with respect to the  $H^{-1}$ -inner product.
- (P2) The spectrum of  $\mathcal{A}$  contains a countable system of real eigenvalues.

(P3) Let  $\{\lambda_n\}_{n \in \mathbb{N}}$  be eigenvalues of  $\mathcal{A}$  with  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ . Then  $\{\lambda_n\}_{n \in \mathbb{N}}$  are characterized by

$$\lambda_1 = - \inf_{w \in \mathcal{E} \setminus \{0\}} \frac{I[w, w]}{(w, w)_{-1}}, \quad \lambda_n = - \sup_{\mathcal{W} \in \Sigma_{n-1}} \inf_{w \in \mathcal{W}^\perp \setminus \{0\}} \frac{I[w, w]}{(w, w)_{-1}}.$$

Here,  $\Sigma_n$  is the class of subspaces of  $\mathcal{E}$  with  $n$ -dimension and  $\mathcal{W}^\perp$  is the orthogonal subspace of  $\mathcal{W}$  with respect to the  $H^{-1}$ -inner product.

(P4) The eigenvalues of  $\mathcal{A}$  depend continuously on  $\kappa_{\Pi_\pm}$ ,  $\kappa_{\Gamma_\star}$ ,  $d$ , and  $\theta_\pm$ , and are monotone decreasing with respect to  $\kappa_{\Pi_\pm}$ .

Concerning proofs, see [5, 8] for (P1) and (P2), and [4, Chapter VI] for (P3) and (P4).

### 3 Criteria of Stability

If the maximal eigenvalue  $\lambda_1$  for (6) is negative, the steady states  $\Gamma_\star$  are linearly stable under an axisymmetric perturbation. First, we show the following lemma.

**Lemma 3.1** Set

$$\Lambda_\pm := \kappa_{\Pi_\pm} \csc \theta_\pm - \kappa_{\Gamma_\star} \cot \theta_\pm.$$

Then there exists  $m > 0$  and  $\delta > 0$  such that

$$I[w, w] > 0 \quad (w \in \mathcal{E} \setminus \{0\}),$$

provided that  $\Lambda_-, \Lambda_+ > m$  and  $d < \delta$ .

Regarding a proof, see [10].

Lemma 3.1 implies that there exist  $m > 0$  and  $\delta > 0$  such that the maximal eigenvalue  $\lambda_1$  is non-positive, provided that  $\kappa_{\Pi_-}, \kappa_{\Pi_+} > m$  and  $d < \delta$ . That is, all of eigenvalues are non-positive. According to (P4), the eigenvalues depend continuously on the parameters and are monotone decreasing with respect to  $\kappa_{\Pi_\pm}$ . Thus we want to know the condition for the parameters that the zero is an eigenvalue for the eigenvalue problem (6). Now we consider the zero-eigenvalue problem

$$\Delta_{\Gamma_\star} L[w] = 0 \quad \text{for } s \in [0, d], \tag{7}$$

$$\partial_s w \pm (\kappa_{\Pi_\pm} \csc \theta_\pm - \kappa_{\Gamma_\star} \cot \theta_\pm) w = 0 \quad \text{at } s = 0, d, \tag{8}$$

$$\partial_s L[w] = 0 \quad \text{at } s = 0, d. \tag{9}$$

Multiplying  $L[w]$  by the both side of (7) and integrating it by parts with (9), we have

$$\|\partial_s L[w]\|_{L^2(\Gamma_\star)}^2 = 0.$$

Hence  $L[w]$  must be constants, so that we can obtain the solutions of (7) satisfying the boundary condition (9) if we solve

$$L[w] = 0, \quad L[w] = \gamma (\neq 0). \quad (10)$$

Let  $w_1, w_2$  be fundamental solutions of  $L[v] = 0$  and  $w_3$  be a solution of  $L[v] = \gamma$ . Then a solution of (7) satisfying the boundary condition (9) is represented by

$$w(s) = c_1 w_1(s) + c_2 w_2(s) + c_3 w_3(s), \quad (11)$$

where  $c_i (i = 1, 2, 3)$  are arbitrary constants. Deriving the condition of parameters that  $w$  given by (11) is a non-trivial solution satisfying the boundary condition (8) and

$$\int_0^d v y_* ds = 0,$$

it gives the condition of parameters that the zero is an eigenvalue for (6). That is, the zero is an eigenvalue if and only if the parameters satisfy

$$\begin{vmatrix} w'_1(0) - \Lambda_- w_1(0) & w'_2(0) - \Lambda_- w_2(0) & w'_3(0) - \Lambda_- w_3(0) \\ w'_1(d) + \Lambda_+ w_1(d) & w'_2(d) + \Lambda_+ w_2(d) & w'_3(d) + \Lambda_+ w_3(d) \\ \int_0^d w_1 y_* ds & \int_0^d w_2 y_* ds & \int_0^d w_3 y_* ds \end{vmatrix} = 0, \quad (12)$$

where  $\Lambda_{\pm} = \kappa_{\Pi_{\pm}} \csc \theta_{\pm} - \kappa_{\Gamma_*} \cot \theta_{\pm}$ . Setting

$$\mathbf{w}(s) = (w_1(s), w_2(s), w_3(s))^T, \quad \mathbf{I}(d) = \left( \int_0^d w_1 y_* ds, \int_0^d w_2 y_* ds, \int_0^d w_3 y_* ds \right)^T,$$

(12) is equivalent to

$$A^w \kappa_{\Pi_-} \kappa_{\Pi_+} + B_-^w \kappa_{\Pi_-} + B_+^w \kappa_{\Pi_+} + C^w = 0, \quad (13)$$

where

$$\begin{aligned} A^w &= -(\mathbf{w}(0) \times \mathbf{w}(d), \mathbf{I}(d))_{\mathbb{R}^3}, \\ B_-^w &= \{-(\mathbf{w}(0) \times \mathbf{w}'(d), \mathbf{I}(d))_{\mathbb{R}^3} + (\mathbf{w}(0) \times \mathbf{w}(d), \mathbf{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(d) \cot \theta_+\} \sin \theta_+, \\ B_+^w &= \{(\mathbf{w}'(0) \times \mathbf{w}(d), \mathbf{I}(d))_{\mathbb{R}^3} + (\mathbf{w}(0) \times \mathbf{w}(d), \mathbf{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(0) \cot \theta_-\} \sin \theta_-, \\ C^w &= \{(\mathbf{w}'(0) \times \mathbf{w}'(d), \mathbf{I}(d))_{\mathbb{R}^3} \\ &\quad - (\mathbf{w}(0) \times \mathbf{w}(d), \mathbf{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(d) \kappa_{\Gamma_*}(0) \cot \theta_+ \cot \theta_-\} \sin \theta_+ \sin \theta_-. \end{aligned}$$

Then we obtain the following three representations of (13).

(I) If  $A^w \neq 0$  and  $B_-^w B_+^w - A^w C^w \neq 0$ ,

$$(13) \quad \Leftrightarrow \quad \kappa_{\Pi_+} = -\frac{B_-^w}{A^w} + \frac{\frac{B_-^w B_+^w - A^w C^w}{(A^w)^2}}{\kappa_{\Pi_-} - \left(-\frac{B_+^w}{A^w}\right)}.$$



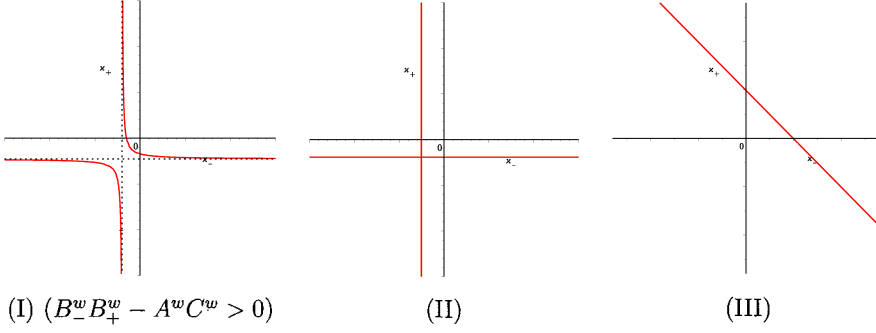


Figure 5: The configurations of (I), (II), and (III).

(II) If  $A^w \neq 0$  and  $B_-^w B_+^w - A^w C^w = 0$ ,

$$(13) \Leftrightarrow \left\{ \kappa_{\Pi_-} - \left( -\frac{B_+^w}{A^w} \right) \right\} \left\{ \kappa_{\Pi_+} - \left( -\frac{B_-^w}{A^w} \right) \right\} = 0.$$

(III) If  $A^w = 0$ ,

$$(13) \Leftrightarrow B_-^w \kappa_{\Pi_-} + B_+^w \kappa_{\Pi_+} + C^w = 0.$$

The coefficients  $A^w$ ,  $B_\pm^w$ , and  $C^w$  depend on the configurations of the steady states  $\Gamma_*$ . Thus, let us derive  $w_i$  when  $\Gamma_*$  are the Delaunay surfaces with non-zero constant mean curvature. Since the generating curves  $(x_*(s), y_*(s))^T$  are given by (4), the coefficient  $|A_*|^2$  in the operator  $L[w]$  and  $\kappa_{\Gamma_*}$  in the boundary condition (8) are

$$|A_*|^2 = \frac{4H_*^2 \{B^2(B - \sin(2H_*(s - \tau)))^2 + (1 - B \sin(2H_*(s - \tau)))^2\}}{(1 + B^2 - 2B \sin(2H_*(s - \tau)))^2},$$

$$\kappa_{\Gamma_*} = \frac{2BH_*(B - \sin(2H_*(s - \tau)))}{1 + B^2 - 2B \sin(2H_*(s - \tau))}.$$

Solving  $L[w] = 0$  and  $L[w] = 1$  (we choose 1 as  $\gamma$  in (10)), we obtain

$$\begin{cases} w_1(s) = \frac{\cos(2H_*(s - \tau))}{\sqrt{1 + B^2 - 2B \sin(2H_*(s - \tau))}}, \\ w_2(s) = \sin(2H_*(s - \tau)) + 2H_* \left\{ \frac{1 + B^2}{2} I_1(s) - \frac{1}{2} I_2(s) \right\}, \\ w_3(s) = \frac{1}{4H_*^2} + \frac{B}{2H_*} I_1(s) w_1(s), \end{cases} \quad (14)$$

where

$$I_1(s) = I_1(s; H_*, B, \tau) := \int_0^s \frac{1}{\sqrt{1 + B^2 - 2B \sin(2H_*(\sigma - \tau))}} d\sigma,$$

$$I_2(s) = I_2(s; H_*, B, \tau) := \int_0^s \sqrt{1 + B^2 - 2B \sin(2H_*(\sigma - \tau))} d\sigma.$$

Set

$$H_*^+ = -H_* (> 0), \quad \alpha = H_*^+ \tau + \frac{\pi}{4}, \quad \beta = H_*^+ \tau - \frac{\pi}{4}$$

and let  $\alpha \in [-\pi/2, \pi/2)$ ,  $\beta < 0$ , and

$$\begin{cases} -\frac{\pi}{2} + m\pi < H_*^+ s - \alpha < -\frac{\pi}{2} + (m+1)\pi & (m \in \mathbb{N} \cup \{0\}) \text{ for } B \neq 1, \\ 0 < H_*^+ s - \beta < \pi & \text{ for } B = 1. \end{cases}$$

Then  $I_1(s; -H_*^+, B, \tau)$  and  $I_2(s; -H_*^+, B, \tau)$  are represented by

$$\begin{aligned} & I_1(s; -H_*^+, B, \tau) \\ &= \begin{cases} \frac{1}{H_*^+(1+B)} \{2mK(k) + (-1)^m F(\sin(H_*^+ s - \alpha); k) - F(\sin(-\alpha); k)\} & (B \neq 1), \\ \frac{1}{2H_*^+} \left\{ \log \left( \tan \left( \frac{H_*^+ s - \beta}{2} \right) \right) - \log \left( \tan \left( -\frac{\beta}{2} \right) \right) \right\} & (B = 1), \end{cases} \\ & I_2(s; -H_*^+, B, \tau) \\ &= \begin{cases} \frac{1+B}{H_*^+} \{2mE(k) + (-1)^m E(\sin(H_*^+ s - \alpha); k) - E(\sin(-\alpha); k)\} & (B \neq 1), \\ \frac{2}{H_*^+} \{\cos \beta - \cos(H_*^+ s - \beta)\} & (B = 1), \end{cases} \end{aligned}$$

where  $k = 2\sqrt{B}/(1+B)$ ,  $K(k)$  and  $E(k)$  are complete elliptic integrals of the 1st and 2nd kind, and  $F(\eta; k)$  and  $E(\eta; k)$  are incomplete elliptic integrals of the 1st and 2nd kind. In this paper, the elliptic integrals are given by

$$K(k) = \int_0^1 \frac{1}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}} d\xi, \quad E(k) = \int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}} d\xi,$$

$$F(\eta; k) = \int_0^\eta \frac{1}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}} d\xi, \quad E(\eta; k) = \int_0^\eta \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}} d\xi.$$

Substituting (14) for (13), we are led to

$$\begin{aligned} & A^D(H_*^+, B, d, \tau) \kappa_{\Pi-} \kappa_{\Pi+} + B_-^D(H_*^+, B, d, \tau, \theta_+) \kappa_{\Pi-} + B_+^D(H_*^+, B, d, \tau, \theta_-) \kappa_{\Pi+} \\ & + C^D(H_*^+, B, d, \tau, \theta_+, \theta_-) = 0. \end{aligned} \tag{15}$$

The precise forms of  $A^D$ ,  $B_\pm^D$ , and  $C^D$  are obtained by using Maple 17. Here, we show only the form of  $A^D$ :

$$\begin{aligned}
A^D(H_*^+, B, d, \tau) &= \frac{1}{8(H_*^+)^3 P Q} \left[ (H_*^+)^2 (1 - B^2)^2 I_1^2 \cos(2H_*^+ \tau) \cos(2H_*^+ (d - \tau)) \right. \\
&\quad - 4(H_*^+)^2 (1 + B^2) I_1 I_2 \cos(2H_*^+ \tau) \cos(2H_*^+ (d - \tau)) \\
&\quad + 3(H_*^+)^2 I_2^2 \cos(2H_*^+ \tau) \cos(2H_*^+ (d - \tau)) \\
&\quad + 2H_*^+ (1 + B^2) I_1 \{ P \sin(2H_*^+ \tau) \cos(2H_*^+ (d - \tau)) + Q \cos(2H_*^+ \tau) \sin(2H_*^+ (d - \tau)) \} \\
&\quad - 4H_*^+ B I_1 \{ P \cos(2H_*^+ (d - \tau)) - Q \cos(2H_*^+ \tau) \} \\
&\quad - 4H_*^+ I_2 \{ P \sin(2H_*^+ \tau) \cos(2H_*^+ (d - \tau)) + Q \cos(2H_*^+ \tau) \sin(2H_*^+ (d - \tau)) \} \\
&\quad + 2PQ \{ 1 + \sin(2H_*^+ \tau) \sin(2H_*^+ (d - \tau)) \} \\
&\quad \left. - (P^2 + Q^2) \cos(2H_*^+ \tau) \cos(2H_*^+ (d - \tau)) \right],
\end{aligned}$$

where

$$\begin{aligned}
P(H_*^+, B, \tau) &= \sqrt{1 + B^2 - 2B \sin(2H_*^+ \tau)}, \\
Q(H_*^+, B, d, \tau) &= \sqrt{1 + B^2 + 2B \sin(2H_*^+ (d - \tau))}.
\end{aligned}$$

Moreover, by the help with Maple 17, we have

$$\begin{aligned}
&B_-^D(H_*^+, B, d, \tau, \theta_+) B_+^D(H_*^+, B, d, \tau, \theta_-) - A^D(H_*^+, B, d, \tau) C^D(H_*^+, B, d, \tau, \theta_+, \theta_-) \\
&= \frac{1}{16(H_*^+)^4 P Q} \left[ H_*^+ \{ (1 + B^2) (1 + \sin(2H_*^+ \tau) \sin(2H_*^+ (d - \tau))) - (P^2 + Q^2) \} I_1 \right. \\
&\quad + H_*^+ (3 - \sin(2H_*^+ (d - \tau)) \sin(2H_*^+ \tau)) I_2 \\
&\quad \left. - P \cos(2H_*^+ \tau) \sin(2H_*^+ (d - \tau)) - Q \sin(2H_*^+ \tau) \cos(2H_*^+ (d - \tau)) \right]^2 \geq 0.
\end{aligned}$$

**Theorem 3.1** Set

$$\begin{aligned}
&D(\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm) \\
&:= A^D(H_*^+, B, d, \tau) \kappa_{\Pi_-} \kappa_{\Pi_+} + B_-^D(H_*^+, B, d, \tau, \theta_+) \kappa_{\Pi_-} + B_+^D(H_*^+, B, d, \tau, \theta_-) \kappa_{\Pi_+} \\
&\quad + C^D(H_*^+, B, d, \tau, \theta_+, \theta_-),
\end{aligned}$$

and let  $q_1$  be the value of  $H_*^+ d$  which is the 1st zero-point of  $A^D$ . If the parameters  $\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm$  satisfy

$$\hat{D}(\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm) > 0, \quad \kappa_{\Pi_-} > -\frac{B_+^D(H_*^+, B, d, \tau, \theta_-)}{A^D(H_*^+, B, d, \tau)}, \quad \text{and} \quad H_*^+ d < q_1, \quad (16)$$

then the Delaunay surfaces are linearly stable under an axisymmetric perturbation.

**Theorem 3.2** If  $H_*^+ d \geq q_1$ , then there are no pairs of  $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$  such that the Delaunay surfaces are stable.

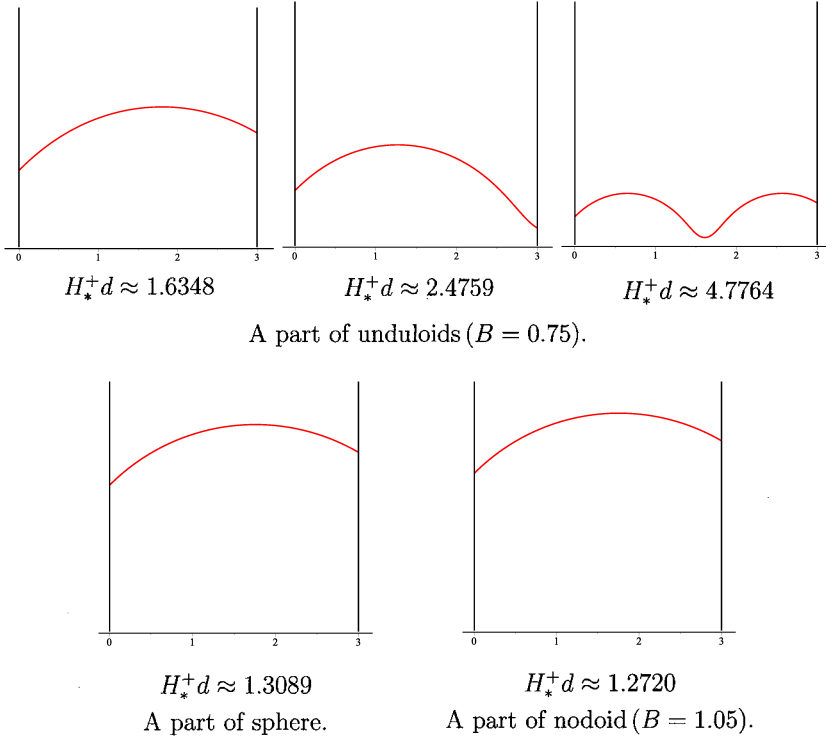


Figure 6: The Delaunay surfaces with  $\theta_- = \frac{\pi}{4}$  and  $\theta_+ = \frac{\pi}{3}$ .

## 4 Examples

Concerning criteria of stability for cylinders and unduloids with  $\tau = \pi/(4H_*^+)$  under  $\theta_{\pm} = \pi/2$ , see [10, 11]. In this paper, we consider the stability of unduloids, sphere, and nodoid given by Fig. 6.

For unduloids in this setting, we can obtain  $q_1 \approx 2.6310$  by the help with Maple 17. Thus, by Theorem 3.2, the unduloid with  $H_*^+ d \approx 4.7764$  is unstable. In the cases  $H_*^+ d \approx 1.6348$  and  $H_*^+ d \approx 2.4759$ , the criteria of the unduloids are given by Fig. 7. By Theorem 3.1, unduloids are stable under an axisymmetric perturbation, provided that  $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$  is included in the gray parts in Fig. 7. For  $H_*^+ d \approx 1.6348$ ,  $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0, 0)$  is included in the gray part, so that the unduloid with  $H_*^+ d \approx 1.6348$  is stable under an axisymmetric perturbation. On the other hand, for  $H_*^+ d \approx 2.4759$ ,  $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0, 0)$  is not included in the gray part. Thus the unduloid with  $H_*^+ d \approx 2.4759$  is unstable.

For sphere in this setting, we consider the problem in the interval  $[0, 2.3561]$ . In this

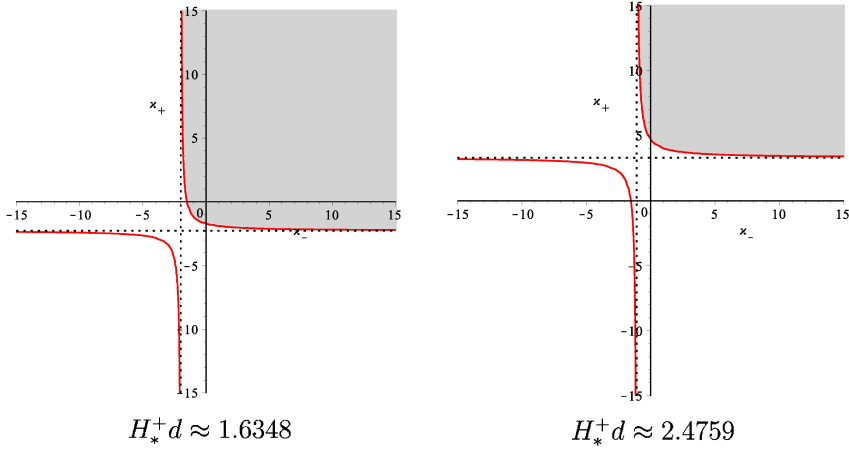


Figure 7: The criteria of unduloids with  $H_*^+ d \approx 1.6348$  and  $H_*^+ d \approx 2.4759$ .

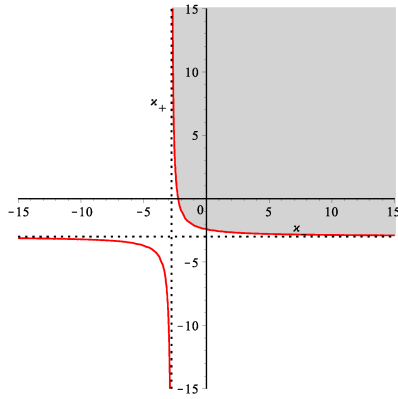


Figure 8: The criterion of the sphere with  $H_*^+ d \approx 1.3089$ .

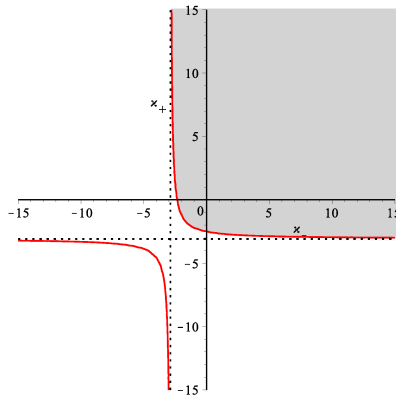


Figure 9: The criterion of the nodoid with  $H_*^+ d \approx 1.2720$ .

interval, we have no value of  $H_*^+ d$  which is the zero-point of  $A^D$ . Thus we can judge the stability by using Fig. 8.  $(\kappa_{\Pi-}, \kappa_{\Pi+}) = (0, 0)$  is included in the gray part, so that the sphere with  $H_*^+ d \approx 1.3089$  is stable under an axisymmetric perturbation.

For nodoid in this setting, we can obtain  $q_1 \approx 2.3389$  by the help with Maple 17. The criterion of the nodoid with  $H_*^+ d \approx 1.2720$  is given by Fig. 9. Then we see that  $(\kappa_{\Pi-}, \kappa_{\Pi+}) = (0, 0)$  is included in the gray part. Thus the nodoid with  $H_*^+ d \approx 1.2720$  is stable under an axisymmetric perturbation.

**Acknowledgements** This work was supported by JSPS KAKENHI Grant Numbers 24540200, 24244012, 25247008.

## References

- [1] H. Abels, H. Garcke, and L. Müller, *Stability of spherical caps under the volume-preserving mean curvature flow with line tension*, Nonlinear Anal., 117 (2015), 8–37.
- [2] M. Athanassenas, *A variational problem for constant mean curvature surfaces with free boundary*, J. Reine Angew. Math., 377 (1987), 97–107.
- [3] A. J. Bernoff, A. L. Bertozzi, T. P. Witelski, *Axisymmetric surface diffusion: dynamics and stability of self-similar pinchoff*, J. Statist. Phys., 93 (1998), no. 3-4, 725–776.
- [4] R. Courant and D. Hilbert, *Methods of mathematical physics*, vol.I, Interscience, New York, 1953.
- [5] D. Depner, *Linearized stability analysis of surface diffusion for hypersurfaces with boundary contact*, Math. Nachr., 285 (2012), no. 11-12, 1385–1403.

- [6] L. G. Fel and B. Y. Rubinstein, *Stability of axisymmetric liquid bridges*, Z. Angew. Math. Phys., 66 (2015), Issue 6, 3447–3471.
- [7] R. Finn, *Equilibrium Capillary Surfaces*, Grundlehren der mathematischen Wissenschaften 284, Springer New York, 1986.
- [8] H. Garcke, K. Ito, and Y. Kohsaka, *Surface diffusion with triple junctions: a stability criterion for stationary solutions*, Adv. Differential Equations, 15 (2010), no. 5-6, 437–472.
- [9] Katsuei Kenmotsu, *Surfaces with Constant Mean Curvature*, Translations of Mathematical Monographs, AMS, 2003.
- [10] Yoshihito Kohsaka, *Stability analysis of Delaunay surfaces as steady states for the surface diffusion equation*, Geometric Properties for Parabolic and Elliptic PDE's, Springer Proceedings in Mathematics & Statistics 176, 121–148, Springer International Publishing Switzerland, 2016.
- [11] Yoshihito Kohsaka, *On the criteria for the stability of unduloids*, To appear in RIMS Kôkyûroku Bessatsu B63 (2017), 167–192.
- [12] J. LeCrone, G. Simonett, *On well-posedness, stability, and bifurcation for the axisymmetric surface diffusion flow*, SIAM J. Math. Anal., 45 (2013), no. 5, 2834–2869.
- [13] A. D. Myshkis, V. G. Babskii, N. D. Kopachevskii, L. A. Slobozhanin, and A. D. Tyuptsov, *Low-Gravity Fluid Mechanics*, Springer-Verlag Berlin Heidelberg, 1987.
- [14] B. Y. Rubinstein and L. G. Fel, *Stability of unduloidal and nodoidal menisci between two solid spheres*, J. Geom. Symmetry Phys., 39 (2015), 77–98.
- [15] T. I. Vogel, *Stability of a liquid drop trapped between two parallel planes*, SIAM J. Appl. Math., 47 (1987), no. 3, 516–525.
- [16] T. I. Vogel, *Stability of a liquid drop trapped between two parallel planes. II. General contact angles*, SIAM J. Appl. Math., 49 (1989), no. 4, 1009–1028.
- [17] T. I. Vogel, *Convex, rotationally symmetric liquid bridges between spheres*, Pacific J. Math., 224 (2006), no. 2, 367–377.
- [18] T. I. Vogel, *Liquid Bridges Between Balls: The Small Volume Instability*, J. Math. Fluid Mech., 15 (2013), no. 2, 397–413.
- [19] T. I. Vogel, *Liquid Bridges between Contacting Balls*, J. Math. Fluid Mech., 16 (2014), 737–744.
- [20] S. Yotsutani and M. Murai, 楕円関数と仲良くなろう, 日本評論社, 2013.